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Generalized V-univexity type-I for multiobjective programming with n-set functions

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Abstract In the last time important results in multiobjective programming involving type-I functions were obtained (Yuan et al. in: Konnov et al. (eds) Lecture notes in economics and mathematical systems, 2007; Mishra et al. An Univ Bucureşti Ser Mat, LII(2): 207–224, 2003). Following one of these ways, we study optimality conditions and generalized Mond-Weir duality for multiobjective programming involving n-set functions which satisfy appropriate generalized univexity V-type-I conditions. We introduce classes of functions called (ρ, ρ')-V-univex type-I, (ρ, ρ')-quasi V-univex type-I, (ρ, ρ')-pseudo V-univex type-I, (ρ, ρ')-quasi pseudo V-univex type-I, and (ρ, ρ')-pseudo quasi V-univex type-I. Finally, a general frame for constructing functions of these classes is considered.

Keywords Optimality conditions · Duality · Multi objective programming · n-set functions · Generalized univexity V-type-I

1 Introduction

The analysis of optimization problems with set or n-set functions i.e. selection of measurable subsets from a given space, has been the subject of several papers [2–5, 14, 16, 18–22].

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These problems occur naturally in a variety of situations dealing with optimal selection of measurable subsets.

The concept of optimizing set functions arises in various mathematical areas. Thus, an early result was the Neyman–Pearson lemma of statistics [17], which states only a sufficient condition for maximizing an integral over a single set. Later, it was established the necessity of this condition and the existence of a solution.

A generalization of these results to n-set and a duality theory was developed in [6,7], for regional design problems (districting, facility location, warehouse layout, urban planning).

Also, constrained optimization problems involving set or n-set functions have arisen in fluid flow, electrical insulator design and optimal plasma confinement [3,4].

General theory for optimizing n-set functions was first developed by Morris [16] who, for functions of a single set, obtained results that are similar to the standard mathematical programming problem. Zalmai [28] considered several practical applications for a class of nonlinear programming problems involving a single objective and differentiable n-set functions, and established sufficient optimality and duality results under generalized ρ-convexity conditions. Many publications have appeared in the last decade dealing with duality in multi-objective programming involving set-functions [2,8,14,18,19,22,23,27], and univexity [10–12,25].

For a historical survey of optimality conditions and duality for programming problems involving set and n-set functions the reader is referred to [24].

Recently, important results in multiobjective programming involving type-I functions were obtained. Thus, Yuan et al. [26] introduce the class of (C, α, ρ,d) type-I functions, which extends many known classes of maps.

Mishra et al. [14] extended the generalized type-I vector valued functions introduced by Aghezzaf and Hachimi [1] to n-set functions and they established optimality conditions and Mond–Weir type, respectively general Mond–Weir type duality results for a multiobjective programming problem involving n-set functions.

In [20], Preda and Bătătorescu considered a minmax programming problem involving several generalized B-vex n-set functions and obtained optimality results and Wolfe type duality theorems.

Along the line of Jeyakumar and Mond [9], Preda and Stancu-Minasian [21] defined new classes of n-set functions, called d-type-I, d-quasi type-I, d-pseudo type-I, d-quasi-pseudo type-I, d-pseudo-quasi type-I. They obtained necessary and sufficient optimality criteria and duality results, considering the concept of a weak minimum. Mishra et al. [13] established for multiobjective programming problems some sufficiency results using Lagrange multiplier conditions and under various types of generalized V-univexity type-I requirements they proved weak, strong and converse duality theorems.

Following this way, in Sect. 2 of this paper we consider some classes of univexity type-I functions and we state sufficient and necessary optimality condition for a multi-objective optimization problem involving set functions. In Sect. 3 we give some duality results for a generalized Mond–Weir dual problem and we prove weak, strong and converse duality theorems. In the last section we give a procedure, through which we may transform examples of V-univexity type optimization problems on $\mathbb{R}^n$ into V-univexity type optimization problems that involve set or n-set functions.

2 Preliminaries

In this section we introduce the notations and definitions which will be used throughout the paper. Let $\mathbb{R}^n$ be the n-dimensional Euclidean space.

For any vectors $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $y = (y_1, \dots, y_n) \in \mathbb{R}^n$, we use the following notation:

$$x < y \quad \text{iff} \quad x_i < y_i\,,\ i = 1, 2, \dots, n; \quad x \leqq y \quad \text{iff} \quad x_i \leqq y_i\,,\ i = 1, 2, \dots, n;$$

$$x \leq y \quad \text{iff} \quad x \leqq y, \text{ but } x \neq y; \quad x^\top y = \sum_{i=1}^{n} x_i y_i.$$

Let $\mathbb{R}^n_+$ be the nonnegative orthant of $\mathbb{R}^n$, i.e.

$$\mathbb{R}^n_+ = \left\{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i \geqq 0,\ i = 1, \dots, n\right\}.$$

For an arbitrary vector $x \in \mathbb{R}^n$ and a subset J of the index set $\{1, 2, \dots, n\}$, we denote by x_J the vector with components x_j, $j \in J$.

Let (X, Γ, μ) be a finite atomless measure space with $L_1(X, \Gamma, \mu)$ a separable space. For $h \in L_1(X, \Gamma, \mu)$ and $Z \in \Gamma$ with indicator function $I_Z \in L_\infty(X, \Gamma, \mu)$, the integral $\int_Z h\mathrm{d}\mu$ will be denoted by $\langle h, I_Z\rangle$.

Now we shall define the notion of differentiability for n-set functions. Morris [16] introduced differentiability for set functions, subsequently extended by Corley [5] to n-set functions.

A function $\varphi : \Gamma \to \mathbb{R}$ is said to be differentiable at T if there is $D\varphi_T \in L_1(X, \Gamma, \mu)$, called the derivative of φ at T, such that

$$\varphi(S) = \varphi(T) + \langle D\varphi_T, I_S - I_T\rangle + \psi(S, T)$$

for each $S \in \Gamma$, where $\psi : \Gamma \times \Gamma \to \mathbb{R}$ and has the property that $\psi(S, T)$ is $o(d'(S, T))$, that is, $\lim_{d'(S,T)\to 0} \psi(S, T)/d'(S, T) = 0$, and d' is a pseudometric on Γ [16].

A function $h : \Gamma^n \to \mathbb{R}$ is said to have a partial derivative at $S^0 = \left(S_1^0, \dots, S_n^0\right)$ with respect to its k-th argument, $1 \leqq k \leqq n$, if the function

$$\varphi(S_k) = h(S_1^0, \dots, S_{k-1}^0, S_k, S_{k+1}^0, \dots, S_n^0)$$

has derivative $D\varphi_{S_k^0}$, and we define $D_k h(S^0) = D\varphi_{S_k^0}$. If the $D_k h(S^0)$, $1 \leqq k \leqq n$, all exist, then we put $Dh(S^0) = (D_1 h(S^0), \dots, D_n h(S^0))$. If $H : \Gamma^n \to \mathbb{R}^m$, $H = (H_1, \dots, H_m)$, we put $D_k H(S^0) = (D_k H_1(S^0), \dots, D_k H_m(S^0))$.

A function $h : \Gamma^n \to \mathbb{R}$ is said to be differentiable at $S^0 \in \Gamma^n$ if there is $Dh(S^0)$ and $\psi : \Gamma^n \times \Gamma^n \to \mathbb{R}$ such that

$$h(S) = h(S^0) + \sum_{k=1}^{n} \left\langle D_k h(S^0), I_{S_k} - I_{S_k^0}\right\rangle + \psi(S, S^0)$$

where $\psi(S, S^0)$ is $o[d(S, S^0)]$ for all $S \in \Gamma^n$ and here d is a pseudometric on Γ^n, i.e., $d(S, T) = \left[\sum_{k=1}^{n} \mu^2(S_k \Delta T_k)\right]^{1/2}$, where Δ denotes the symmetric difference [5].

A vector valued set function $f = \left(f_1, \dots, f_p\right) : \Gamma \to \mathbb{R}^p$ is differentiable on Γ if its all component functions f_i, $1 \leqq i \leqq p$, are differentiable on Γ.

In this paper we consider the n-set function multiobjective optimization problem

$$\text{minimize}\left\{f(S) = \left(f_1(S), \dots, f_p(S)\right) \mid g(S) \leqq 0,\ S \in \Gamma^n\right\}, \tag{VP}$$

where $f : \Gamma^n \to \mathbb{R}^p$ and $g : \Gamma^n \to \mathbb{R}^m$, $g = (g_1, \dots, g_m)$, are differentiable n-set functions defined on Γ^n. The problem is to find the collection of (properly) efficient sets defined below.

Let $\mathcal{P} = \left\{S \in \Gamma^n \mid g(S) \leqq 0\right\}$ be the set of all feasible solutions to problem (VP).

Definition 1 A feasible solution $S^0 \in \mathcal{P}$ is said to be an efficient solution (Pareto solution) to problem (VP) if there exists no other feasible solution $S \in \mathcal{P}$ such that $f(S) \leq f(S^0)$.

Definition 2 An efficient solution S^0 to (VP) is called properly efficient (Geoffrion solution), if there is a positive number M with the property that, if $f_i(S) < f_i(S^0)$ for any i and $S \in \mathcal{P}$, then $\dfrac{f_i(S^0) - f_i(S)}{f_j(S) - f_j(S^0)} \leqq M$ for some j for which $f_j(S) > f_j(S^0)$.

We shall consider a partition $\{J_0, J_1, \ldots, J_k\}$ of the index set $\{1, 2, \ldots, m\}$, that is, $\bigcup_{s=0}^{k} J_s = \{1, 2, \ldots, m\}$ and $J_s \cap J_t = \varnothing$, for any $s \neq t$.

Put

$$\psi_i\left(S, \lambda_{J_0}\right) = f_i(S) + \lambda_{J_0}^{\top} g_{J_0}(S)$$

for any i, $1 \leqq i \leqq p$, where $\lambda \in \mathbb{R}_+^m$ is a given vector. Moreover, we consider vectors $\rho = \left(\rho_1, \ldots, \rho_p\right) \in \mathbb{R}^p$, $\rho' = \left(\rho_1', \ldots, \rho_k'\right) \in \mathbb{R}^k$, and real numbers $\rho_0, \rho_0' \in \mathbb{R}$.

The following definitions extend similar concepts defined by Jeyakumar and Mond [9] and Mishra et al. [13].

Definition 3 We say that problem (VP) is (ρ, ρ')-V-univex type I at $S^0 \in \mathcal{P}$ according to the partition $\{J_0, J_1, \ldots, J_k\}$ relative to $\lambda \in \mathbb{R}_+^m$, if there is positive real functions $\alpha_1, \ldots, \alpha_p$ and $\beta_1, \ldots, \beta_k$ defined on $\Gamma^n \times \Gamma^n$, nonnegative functions b_0 and b_1, also defined on $\Gamma^n \times \Gamma^n$, $\varphi_0 : \mathbb{R} \to \mathbb{R}$, $\varphi_1 : \mathbb{R} \to \mathbb{R}$, such that

$$b_0(S, S^0)\varphi_0\left[\psi_i\left(S, \lambda_{J_0}\right) - \psi_i\left(S^0, \lambda_{J_0}\right)\right] \geqq \alpha_i(S, S^0) \sum_{t=1}^{n} \left\langle D_t \psi_i\left(S^0, \lambda_{J_0}\right), I_{S_t} - I_{S_t^0}\right\rangle + \rho_i d^2(S, S^0) \tag{1}$$

and

$$-b_1(S, S^0)\varphi_1\left[\sum_{j \in J_s} \lambda_j g_j(S^0)\right] \geqq \beta_s(S, S^0) \sum_{j \in J_s} \lambda_j \sum_{t=1}^{n} \left\langle D_t g_j(S^0), I_{S_t} - I_{S_t^0}\right\rangle + \rho_s' d^2(S, S^0) \tag{2}$$

for any $S \in \mathcal{P}$, $i \in \{1, \ldots, p\}$, and $s \in \{1, \ldots, k\}$.

If (VP) is (ρ, ρ')-V-univex type I at all $S^0 \in \mathcal{P}$, according to the partition $\{J_0, J_1, \ldots, J_k\}$ relative to $\lambda \in \mathbb{R}_+^m$, then we say that (VP) is (ρ, ρ')-V-univex type I on $\mathcal{P}$ according to the partition $\{J_0, J_1, \ldots, J_k\}$ relative to $\lambda \in \mathbb{R}_+^m$.

If strict inequality holds in (1) (whenever $S \neq S^0$), then we say that (VP) is (ρ, ρ')-semi strictly V-univex type I at S^0 or on $\mathcal{P}$ according to the partition $\{J_0, J_1, \ldots, J_k\}$ relative to $\lambda \in \mathbb{R}_+^m$, depending on the case.

Definition 4 We say that the problem (VP) is (ρ_0, ρ_0')-quasi V-univex type I at $S^0 \in \mathcal{P}$ according to the partition $\{J_0, J_1, \ldots, J_k\}$ relative to $\tau \in \mathbb{R}_+^p$ and $\lambda \in \mathbb{R}_+^m$ if there is positive real functions $\alpha_1, \ldots, \alpha_p$ and $\beta_1, \ldots, \beta_k$ defined on $\Gamma^n \times \Gamma^n$, nonnegative functions b_0 and b_1, also defined on $\Gamma^n \times \Gamma^n$, $\varphi_0 : \mathbb{R} \to \mathbb{R}$, $\varphi_1 : \mathbb{R} \to \mathbb{R}$, such that the implications

$$b_0(S, S^0)\varphi_0 \left[\sum_{i=1}^{p} \tau_i \alpha_i(S, S^0) \left[\psi_i \left(S, \lambda_{J_0}\right) - \psi_i \left(S^0, \lambda_{J_0}\right) \right] \right] \leqq 0$$
$$\Longrightarrow \sum_{i=1}^{p} \tau_i \sum_{t=1}^{n} \left\langle D_t \psi_i \left(S^0, \lambda_{J_0}\right), I_{S_t} - I_{S_t^0} \right\rangle \leqq -\rho_0 d^2(S, S^0), \quad \forall\, S \in \mathcal{P}, \tag{3}$$

and

$$b_1(S, S^0)\varphi_1 \left[\sum_{s=1}^{k} \beta_s(S, S^0) \sum_{j \in J_s} \lambda_j g_j(S^0) \right] \leqq 0$$
$$\Longrightarrow \sum_{j=1,\ j \notin J_0}^{m} \lambda_j \sum_{t=1}^{n} \left\langle D_t g_j(S^0), I_{S_t} - I_{S_t^0} \right\rangle \leqq -\rho_0' d^2(S, S^0), \quad \forall\, S \in \mathcal{P}, \tag{4}$$

both hold.

If (VP) is (ρ_0, ρ_0')-quasi V-univex type I at all $S^0 \in \mathcal{P}$ according to the partition $\{J_0, J_1, \ldots, J_k\}$ relative to $\tau \in \mathbb{R}_+^p$ and $\lambda \in \mathbb{R}_+^m$, then we say that (VP) is $\left(\rho_0, \rho_0'\right)$-quasi V-univex type I on $\mathcal{P}$ according to the partition $\{J_0, J_1, \ldots, J_k\}$ relative to $\tau \in \mathbb{R}_+^p$ and $\lambda \in \mathbb{R}_+^m$.

If the second (implied) inequality in (3) is strict ($S \neq S^0$), then we say that (VP) is (ρ_0, ρ_0') semi strictly quasi V-univex type I at S^0 or on $\mathcal{P}$ according to the partition $\{J_0, J_1, \ldots, J_k\}$ relative to $\tau \in \mathbb{R}_+^p$ and $\lambda \in \mathbb{R}_+^m$, depending on the case.

Definition 5 We say that the problem (VP) is (ρ_0, ρ_0')-pseudo V-univex type I at $S^0 \in \mathcal{P}$ according to the partition $\{J_0, J_1, \ldots, J_k\}$ relative to $\tau \in \mathbb{R}_+^p$ and $\lambda \in \mathbb{R}_+^m$, if there is positive real functions $\alpha_1, \ldots, \alpha_p$ and $\beta_1, \ldots, \beta_k$ defined on $\Gamma^n \times \Gamma^n$, nonnegative functions b_0 and b_1, also defined on $\Gamma^n \times \Gamma^n$, $\varphi_0 : \mathbb{R} \to \mathbb{R}$, $\varphi_1 : \mathbb{R} \to \mathbb{R}$, such that for all $S \in \mathcal{P}$, the implications

$$\sum_{i=1}^{p} \tau_i \sum_{t=1}^{n} \left\langle D_t \psi_i \left(S^0, \lambda_{J_0}\right), I_{S_t} - I_{S_t^0} \right\rangle \geqq -\rho_0 d^2(S, S^0)$$
$$\Longrightarrow b_0(S, S^0)\varphi_0 \left[\sum_{i=1}^{p} \tau_i \alpha_i(S, S^0) \left[\psi_i \left(S, \lambda_{J_0}\right) - \psi_i \left(S^0, \lambda_{J_0}\right) \right] \right] \geqq 0, \tag{5}$$

and

$$\sum_{j=1,\ j \notin J_0}^{m} \lambda_j \sum_{t=1}^{n} \left\langle D_t g_j(S^0), I_{S_t} - I_{S_t^0} \right\rangle \geqq -\rho_0' d^2(S, S^0)$$
$$\Longrightarrow b_1(S, S^0)\varphi_1 \left[\sum_{s=1}^{k} \beta_s(S, S^0) \sum_{j \in J_s} \lambda_j g_j(S^0) \right] \leqq 0, \tag{6}$$

both hold.

If (VP) is (ρ_0, ρ_0')-pseudo V-univex type I at all $S^0 \in \mathcal{P}$ according to the partition $\{J_0, J_1, \ldots, J_k\}$ relative to $\tau \in \mathbb{R}_+^p$ and $\lambda \in \mathbb{R}_+^m$, then we say that (VP) is (ρ_0, ρ_0')-pseudo

V-univex type I on $\mathcal{P}$ according to the partition $\{J_0, J_1, \ldots, J_k\}$ relative to $\tau \in \mathbb{R}^p_+$ and $\lambda \in \mathbb{R}^m_+$.

If the second (implied) inequality in (5) is strict ($S \neq S^0$), then we say that (VP) is (ρ_0, ρ'_0)-semi strictly pseudo V-univex type I at S^0 or on $\mathcal{P}$ according to the partition $\{J_0, J_1, \ldots, J_k\}$ relative to $\tau \in \mathbb{R}^p_+$ and $\lambda \in \mathbb{R}^m_+$, depending on the case.

If the second (implied) inequalities in (5) and (6) are both strict, then we say that (VP) is (ρ_0, ρ'_0)-strictly pseudo V-univex type I at S^0 or on $\mathcal{P}$, according to the partition $\{J_0, J_1, \ldots, J_k\}$ relative to $\tau \in \mathbb{R}^p_+$ and $\lambda \in \mathbb{R}^m_+$, depending on the case.

Definition 6 We say that the problem (VP) is $\left(\rho_0, \rho'_0\right)$-quasi pseudo V-univex type I at $S^0 \in \mathcal{P}$ according to the partition $\{J_0, J_1, \ldots, J_k\}$ relative to $\tau \in \mathbb{R}^p_+$ and $\lambda \in \mathbb{R}^m_+$ if there are some positive real functions $\alpha_1, \ldots, \alpha_p$ and $\beta_1, \ldots, \beta_k$ defined on $\Gamma^n \times \Gamma^n$, nonnegative functions b_0 and b_1, also defined on $\Gamma^n \times \Gamma^n$, $\varphi_0 : \mathbb{R} \to \mathbb{R}$, $\varphi_1 : \mathbb{R} \to \mathbb{R}$, such that the implications

$$b_0(S, S^0)\varphi_0 \left[\sum_{i=1}^{p} \tau_i \alpha_i(S, S^0) \left[\psi_i \left(S, \lambda_{J_0}\right) - \psi_i \left(S^0, \lambda_{J_0}\right)\right] \right] \leqq 0$$
$$\Longrightarrow \sum_{i=1}^{p} \tau_i \sum_{t=1}^{n} \left\langle D_t \psi_i \left(S^0, \lambda_{J_0}\right), I_{S_t} - I_{S_t^0} \right\rangle \leqq -\rho_0 d^2(S, S^0), \quad \forall\, S \in \mathcal{P}, \tag{7}$$

and

$$\sum_{j=1,\ j \notin J_0}^{m} \lambda_j \sum_{t=1}^{n} \left\langle D_t g_j(S^0), I_{S_t} - I_{S_t^0} \right\rangle \geqq -\rho'_0 d^2(S, S^0)$$
$$\Longrightarrow b_1(S, S^0)\varphi_1 \left[\sum_{s=1}^{k} \beta_s(S, S^0) \sum_{j \in J_s} \lambda_j g_j(S^0) \right] \leqq 0, \quad \forall\, S \in \mathcal{P}, \tag{8}$$

do hold.

If (VP) is (ρ_0, ρ'_0)-quasi pseudo V-univex type I at all $S^0 \in \mathcal{P}$ according to the partition $\{J_0, J_1, \ldots, J_k\}$ relative to $\tau \in \mathbb{R}^p_+$ and $\lambda \in \mathbb{R}^m_+$, then we say that (VP) is (ρ_0, ρ'_0)-quasi pseudo V-univex type I on $\mathcal{P}$ according to the partition $\{J_0, J_1, \ldots, J_k\}$ relative to $\tau \in \mathbb{R}^p_+$ and $\lambda \in \mathbb{R}^m_+$.

If the second (implied) inequality in (8) is strict ($S \neq S^0$), then we say that (VP) is (ρ_0, ρ'_0)-quasi strictly pseudo V-univex type I at S^0 or on $\mathcal{P}$, according to the partition $\{J_0, J_1, \ldots, J_k\}$ relative to $\tau \in \mathbb{R}^p_+$ and $\lambda \in \mathbb{R}^m_+$, depending on the case.

Definition 7 We say that the problem (VP) is (ρ_0, ρ'_0) pseudo quasi V-univex type I at $S^0 \in \mathcal{P}$ according to the partition $\{J_0, J_1, \ldots, J_k\}$ relative to $\tau \in \mathbb{R}^p_+$ and $\lambda \in \mathbb{R}^m_+$ if there are some positive real functions $\alpha_1, \ldots, \alpha_p$ and $\beta_1, \ldots, \beta_k$ defined on $\Gamma^n \times \Gamma^n$, nonnegative functions b_0 and b_1, also defined on $\Gamma^n \times \Gamma^n$, $\varphi_0 : \mathbb{R} \to \mathbb{R}$, $\varphi_1 : \mathbb{R} \to \mathbb{R}$, such that for all $S \in \mathcal{P}$, the implications

$$\sum_{i=1}^{p} \tau_i \sum_{t=1}^{n} \left\langle D_t \psi_i \left(S^0, \lambda_{J_0}\right), I_{S_t} - I_{S_t^0} \right\rangle \geqq -\rho_0 d^2(S, S^0)$$
$$\Longrightarrow b_0(S, S^0)\varphi_0 \left[\sum_{i=1}^{p} \tau_i \alpha_i(S, S^0) \left[\psi_i \left(S, \lambda_{J_0}\right) - \psi_i \left(S^0, \lambda_{J_0}\right)\right] \right] \geqq 0, \tag{9}$$

and

$$b_1(S, S^0)\varphi_1 \left[\sum_{s=1}^{k} \beta_s(S, S^0) \sum_{j\in J_s} \lambda_j g_j(S^0) \right] \geqq 0$$

$$\Longrightarrow \sum_{j=1,\ j\notin J_0}^{m} \lambda_j \sum_{t=1}^{n} \left\langle D_t g_j(S^0), I_{S_t} - I_{S_t^0} \right\rangle \leqq -\rho_0' d^2(S, S^0), \tag{10}$$

do hold.

If (VP) is (ρ_0, ρ_0')-pseudo quasi V-univex type I at all $S^0 \in \mathcal{P}$ according to the partition $\{J_0, J_1, \dots, J_k\}$ relative to $\tau \in \mathbb{R}_+^p$ and $\lambda \in \mathbb{R}_+^m$, then we say that (VP) is (ρ_0, ρ_0')-pseudo quasi V-univex type I on $\mathcal{P}$ according to the partition $\{J_0, J_1, \dots, J_k\}$ relative to $\tau \in \mathbb{R}_+^p$ and $\lambda \in \mathbb{R}_+^m$.

If the second (implied) inequality in (9) is strict ($S \neq S^0$), then we say that (VP) is (ρ_0, ρ_0')-strictly pseudo quasi V-univex type I at S^0 or on $\mathcal{P}$ according to the partition $\{J_0, J_1, \dots, J_k\}$ relative to $\tau \in \mathbb{R}_+^p$ and $\lambda \in \mathbb{R}_+^m$, depending on the case.

3 Some optimality conditions

The following result gives sufficient conditions for an element of $\mathcal{P}$ to be an efficient solution of problem (VP) under generalized type-I conditions with respect to a partition of the constraints.

Theorem 1 (Sufficiency) *Suppose that*

(a1) $S^0 \in \mathcal{P}$;

(a2) *there exist* $\tau^0 \in \mathbb{R}_+^p$, $\sum_{i=1}^{p} \tau_i^0 = 1$, *and* $\lambda^0 \in \mathbb{R}_+^m$ *such that*

(a) *for any* $S \in \mathcal{P}$ *we have*

$$\sum_{i=1}^{p} \tau_i^0 \sum_{t=1}^{n} \left\langle D_t f_i(S^0), I_{S_t} - I_{S_t^0} \right\rangle + \sum_{j=1}^{m} \lambda_j^0 \sum_{t=1}^{n} \left\langle D_t g_j(S^0), I_{S_t} - I_{S_t^0} \right\rangle \geqq 0,$$

(b) *with respect to the partition* $\{J_0, J_1, \dots, J_k\}$ *we have* $\sum_{j\in J_s} \lambda_j^0 g_j(S^0) = 0$ *for any* $s \in \{0, 1, \dots, k\}$;

(a3) *problem* (*VP*) *is* (ρ_0, ρ_0')*-quasi strictly pseudo V-univex type I at* S^0 *with* $\rho_0 + \rho_0' \geq 0$ *according to the partition* $\{J_0, J_1, \dots, J_k\}$ *relative to* τ^0, λ^0.

Further, suppose that, for $r \in \mathbb{R}$, *we have*

$$r \leqq 0 \Longrightarrow \varphi_0(r) \leqq 0 \tag{11}$$

$$\varphi_1(r) < 0 \Longrightarrow r < 0 \tag{12}$$

and

$$b_0(S, S^0) > 0, \quad b_1(S, S^0) > 0, \quad \forall\, S \in \mathcal{P}. \tag{13}$$

Then S^0 *is an efficient solution for* (*VP*).

Proof Suppose that S^0 is not an efficient solution of (VP). Then there is an $S' \in \mathcal{P}$ such that $f\left(S'\right) \leq f(S^0)$. Since $\lambda_{J_0}^{0\top} g_{J_0}(S) \leqq 0$ for $\forall\, S \in \mathcal{P}$ and by hypothesis (a2).b, $\lambda_{J_0}^{0\top} g_{J_0}(S^0) = 0$, it follows that for any $i = 1, \ldots, p$, we have

$$\psi_i\left(S', \lambda_{J_0}^0\right) - \psi_i(S^0, \lambda_{J_0}^0) \leqq f_i\left(S'\right) - f_i(S^0) \leqq 0$$

Since $\tau^0 \in \mathbb{R}_+^p$ and $\alpha_i(S', S^0) > 0$, $i = 1, \ldots, p$, it follows

$$\sum_{i=1}^{p} \tau_i^0 \alpha_i(S', S^0)\left[\psi_i(S', \lambda_{J_0}^0) - \psi_i(S^0, \lambda_{J_0}^0)\right] \leqq 0$$

and using (11) and (13) we get

$$b_0(S', S^0)\varphi_0\left[\sum_{i=1}^{p} \tau_i^0 \alpha_i(S', S^0)\left(\psi_i(S', \lambda_{J_0}^0) - \psi_i(S^0, \lambda_{J_0}^0)\right)\right] \leqq 0. \tag{14}$$

From (14) and (7) it follows that

$$\sum_{i=1}^{p} \tau_i^0 \sum_{t=1}^{n}\left\langle D_t \psi_i(S^0, \lambda_{J_0}^0), I_{S_t'} - I_{S_t^0}\right\rangle \leqq -\rho_0 d^2(S', S^0).$$

Since $\psi_i(S^0, \lambda_{J_0}^0) = f_i(S^0) + \sum\limits_{j \in J_0} \lambda_j^0 g_j(S^0)$ and $\sum\limits_{i=1}^{p} \tau_i^0 = 1$, the last inequality becomes

$$\sum_{i=1}^{p} \tau_i^0 \sum_{t=1}^{n}\left\langle D_t f_i(S^0), I_{S_t'} - I_{S_t^0}\right\rangle + \sum_{j \in J_0} \lambda_j^0 \sum_{t=1}^{n}\left\langle D_t g_j(S^0), I_{S_t'} - I_{S_t^0}\right\rangle \leqq -\rho_0 d^2(S', S^0). \tag{15}$$

By inequality (15) and assumption (a2).a, we have

$$\sum_{j=1,\ j \notin J_0}^{m} \lambda_j^0 \sum_{t=1}^{n}\left\langle D_t g_j(S^0), I_{S_t'} - I_{S_t^0}\right\rangle \geqq \rho_0 d^2(S', S^0).$$

Since $\rho_0 + \rho_0' \geq 0$, it follows

$$\sum_{j=1,\ j \notin J_0}^{m} \lambda_j^0 \sum_{t=1}^{n}\left\langle D_t g_j(S^0), I_{S_t'} - I_{S_t^0}\right\rangle \geqq -\rho_0' d^2(S', S^0). \tag{16}$$

From (16) and assumption (a3), it follows that

$$b_1(S', S^0)\varphi_1\left[\sum_{s=1}^{k} \beta_s(S', S^0) \sum_{j \in J_s} \lambda_j^0 g_j(S^0)\right] < 0. \tag{17}$$

From (17), (13) and (12), we have

$$\sum_{s=1}^{k} \beta_s(S', S^0) \sum_{j \in J_s} \lambda_j^0 g_j(S^0) < 0. \tag{18}$$

On the other hand, by hypotheses (a2).b we have $\sum_{j\in J_s} \lambda_j^0 g_j(S^0) = 0$ for any $s \in \{1, \dots, k\}$, which implies

$$\sum_{s=1}^{k} \beta_s(S', S^0) \sum_{j\in J_s} \lambda_j^0 g_j(S^0) = 0. \tag{19}$$

Equation 19 contradicts inequality (18), and therefore the theorem is proved. □

The next result gives necessary condition for a properly efficient solution of (VP).

Theorem 2 (Necessity) (Zalmai [28]). *Suppose that*

(b1) *S^0 is a properly efficient solution of (VP);*
(b2) *there exists $S^* \in P$ with $g_{M_0}(S^*) < 0$ where $M_0 = \{j \mid g_j(S^0) = 0\}$, such that*

$$g_j(S^0) + \sum_{t=1}^{n} \left\langle D_t g_j(S^0), I_{S_t^*} - I_{S_t^0} \right\rangle < 0, \quad \forall\, j \in \{1, \dots, m\}.$$

Then there exist $\tau^0 \in \mathbb{R}^p$, $\tau^0 > 0$, and $\lambda^0 \in \mathbb{R}_+^m$ such that we have

$$\sum_{i=1}^{p} \tau_i^0 \sum_{t=1}^{n} \left\langle D_t f_i(S^0), I_{S_t} - I_{S_t^0} \right\rangle + \sum_{j=1}^{m} \lambda_j^0 \sum_{t=1}^{n} \left\langle D_t g_j(S^0), I_{S_t} - I_{S_t^0} \right\rangle \geqq 0, \quad \text{for any } S \in \mathcal{P},$$

and

$$\lambda_j^0 g_j(S^0) = 0, \ \ j \in \{1, \dots, m\}.$$

4 Generalized Mond–Weir duality

With respect to the partition $\{J_0, J_1, \dots, J_k\}$ of its constraints, we associate with problem (VP) the multiobjective dual problem:

$$\begin{array}{ll} \text{maximize} & \left(f_1(T) + \lambda_{J_0}^\top g_{J_0}(T), \ \dots \ , f_p(T) + \lambda_{J_0}^\top g_{J_0}(T)\right) \\ \text{subject to} & (T, \tau, \lambda) \in \mathcal{D} \end{array} \tag{GMWD}$$

where

$$\mathcal{D} = \left\{ (T, \tau, \lambda) \;\middle|\; \begin{array}{l} \sum_{i=1}^{p} \tau_i \sum_{t=1}^{n} \left\langle D_t \left(f_i + \lambda_{J_0}^\top g_{J_0}\right)(T), I_{S_t} - I_{T_t} \right\rangle \\ \quad + \sum_{j=1}^{m} \lambda_j \sum_{t=1}^{n} \left\langle D_t g_j(T), I_{S_t} - I_{T_t} \right\rangle \geqq 0, \ \forall\, S \in \Gamma^n \\ \lambda_{J_s}^\top g_{J_s}(T) \geqq 0, \ s = 1, \dots, k, \\ T \in \Gamma^n, \ \tau \in \mathbb{R}_+^p, \ e^\top \tau = 1, \ \lambda \in \mathbb{R}_+^m \end{array} \right\}$$

is the set of feasible solutions of (GMWD), with $e = (1, \dots, 1)^\top \in \mathbb{R}^p$.

Theorem 3 (Weak duality) *Suppose that*

(i1) $S \in \mathcal{P}$;
(i2) $(T, \tau, \lambda) \in \mathcal{D}$ *and* $\tau > 0$;
(i3) *problem* (VP) *is* (ρ_0, ρ_0')*-pseudo quasi V-univex type I at* T *with* $\rho_0 + \rho_0' \geq 0$ *according to the partition* $\{J_0, J_1, \ldots, J_k\}$ *with respect to* τ *and* λ *and some positive functions* α_i, $i \in \{1, \ldots, p\}$, β_s, $s \in \{1, \ldots, k\}$.

Further, assume that for $r \in \mathbb{R}$ *we have*

$$\varphi_0(r) \geqq 0 \implies r \geqq 0 \tag{20}$$

$$r \geqq 0 \implies \varphi_1(r) \geqq 0 \tag{21}$$

and

$$b_0(S, T) > 0, \ b_1(S, T) \geqq 0. \tag{22}$$

Then $f(S) \nleq f(T) + \lambda_{J_0}^{\top} g_{J_0}(T) e$.

Proof By hypothesis (i2), we have $\lambda_{J_s}^{\top} g_{J_s}(T) \geqq 0$, for all $s \in \{1, \ldots, k\}$. Since β_s, $s \in \{1, \ldots, k\}$, are all positive functions, we have

$$\sum_{s=1}^{k} \beta_s(S, T) \lambda_{J_s}^{\top} g_{J_s}(T) \geqq 0. \tag{23}$$

By hypothesis (i3), (21), (22) and (23), it follows that

$$\sum_{j=1,\ j \notin J_0}^{m} \lambda_j \sum_{t=1}^{n} \left\langle D_t g_j(T), I_{S_t} - I_{T_t} \right\rangle \leqq -\rho_0' d^2(S, T). \tag{24}$$

From (24) and hypothesis (i2), we obtain

$$\sum_{i=1}^{p} \tau_i \sum_{t=1}^{n} \left\langle D_t \left(f_i + \lambda_{J_0}^{\top} g_{J_0} \right)(T), I_{S_t} - I_{T_t} \right\rangle \geqq \rho_0' d^2(S, T)$$

and by assumption (i3) we have

$$\sum_{i=1}^{p} \tau_i \sum_{t=1}^{n} \left\langle D_t \left(f_i + \lambda_{J_0}^{\top} g_{J_0} \right)(T), I_{S_t} - I_{T_t} \right\rangle \geqq -\rho_0 d^2(S, T). \tag{25}$$

From (25) and using again hypothesis (i3), we get

$$b_0(S, T) \varphi_0 \left[\sum_{i=1}^{p} \tau_i \alpha_i(S, T) \left[\left(f_i + \lambda_{J_0}^{\top} g_{J_0} \right)(S) - \left(f_i + \lambda_{J_0}^{\top} g_{J_0} \right)(T) \right] \right] \geqq 0. \tag{26}$$

From (26), (20) and (22), it follows

$$\sum_{i=1}^{p} \tau_i \alpha_i(S, T) \left[\left(f_i + \lambda_{J_0}^{\top} g_{J_0} \right)(S) - \left(f_i + \lambda_{J_0}^{\top} g_{J_0} \right)(T) \right] \geqq 0. \tag{27}$$

Assume that $f(S) \leq f(T) + \lambda_{J_0}^{\top} g_{J_0}(T) e$. For $S \in \mathcal{P}$ and $\lambda_{J_0} \geqq 0$ we have $\lambda_{J_0}^{\top} g_{J_0}(S) \leqq 0$. It follows

$$f(S) + \lambda_{J_0}^{\top} g_{J_0}(S) e \leq f(T) + \lambda_{J_0}^{\top} g_{J_0}(T) e$$

and since $\alpha_i > 0$, $\tau > 0$, we get

$$\sum_{i=1}^{p} \tau_i \alpha_i (S, T) \left(f(S) + \lambda_{J_0}^{\top} g_{J_0}(S) e - f(T) - \lambda_{J_0}^{\top} g_{J_0}(T) e \right) < 0,$$

which contradicts (27). Therefore, the theorem is proved. □

Following the line used in the proof of the previous theorem we can get the next duality result by replacing the pseudo quasi V-univexity hypothesis with that of the semi strictly V-univexity one.

Theorem 4 (Weak duality) *Suppose that assumptions* (i1) *and* (i2) *of Theorem 3 hold. We also assume that*

- *(i3′) problem (VP) is (ρ, ρ')-semi strictly V-univex type I at T with $\sum_{i=1}^{p} \frac{\tau_i \rho_i}{\alpha_i(S,T)} + \sum_{s=1}^{k} \frac{\rho_s'}{\beta_s(S,T)} \geq 0$ according to the partition $\{J_0, J_1, \ldots, J_k\}$, with respect to λ and some positive functions α_i^*, $i \in \{1, \ldots, p\}$, β_s^*, $s \in \{1, \ldots, k\}$.*

Further, assume that the functions φ_0 and φ_1 have the properties

$$\varphi_0(r) \geqq 0 \implies r \geqq 0 \tag{28}$$

and

$$r \geqq 0 \implies \varphi_1(r) \geqq 0, \tag{29}$$

with φ_0 linear, and

$$b_0(S, T) > 0, \; b_1(S, T) \geqq 0. \tag{30}$$

Then $f(S) \nleq f(T) + \lambda_{J_0}^{\top} g_{J_0}(T) e$.

Theorem 5 (Strong duality) *Suppose that*

(j1) *S^0 is a properly efficient solution of* (*VP*);
(j2) *there exists $S^* \in \mathcal{P}$ with $g_{M_0}(S^*) < 0$, where $M_0 = \left\{ j \mid g_j(S^0) = 0 \right\}$, such that*

$$g_j(S^0) + \sum_{t=1}^{n} \left\langle D_t g_j(S^0), I_{S_t^*} - I_{S_t^0} \right\rangle < 0, \quad \forall \, j \in \{1, \ldots, m\}.$$

Then there exist $\tau^0 \in \mathbb{R}^p$, $\tau^0 > 0$ and $\lambda^0 \in \mathbb{R}_+^m$ such that $(S^0, \tau^0, \lambda^0) \in \mathcal{D}$ and the objective functions of (*VP*) *and* (*GMWD*) *have the same values at S^0 and (S^0, τ^0, λ^0), respectively. If problem (VP) is $\left(\rho_0, \rho_0'\right)$-pseudo quasi V-univex type I with $\rho_0 + \rho_0' \geq 0$ at all feasible solutions of* (*GMWD*) *according to the partition $\{J_0, J_1, \ldots, J_k\}$, with respect to τ^0, λ^0, and conditions* (20)–(22) *of Theorem 3 are satisfied, then $(S^0, \tau^0, \lambda^0) \in \mathcal{D}$ is an efficient solution to* (*GMWD*).

Proof By Theorem 2, there exist $\tau^0 \in \mathbb{R}^p$, $\tau^0 > 0$, and $\lambda^0 \in \mathbb{R}_+^m$ such that, for any $S \in \mathcal{P}$ we have

$$\sum_{i=1}^{p} \tau_i^0 \sum_{t=1}^{n} \left\langle D_t f_i(S^0), I_{S_t} - I_{S_t^0} \right\rangle + \sum_{j=1}^{m} \lambda_j^0 \sum_{t=1}^{n} \left\langle D_t g_j(S^0), I_{S_t^*} - I_{S_t^0} \right\rangle \geqq 0$$

and

$$\lambda_j^0 g_j(S^0) = 0,\ \ j \in \{1, \ldots, m\}\,.$$

Therefore $(S^0, \tau^0, \lambda^0) \in \mathcal{D}$, and obviously both problems (VP) and (GMWD) have the same value of the objective function.

Suppose that (S^0, τ^0, λ^0) is not an efficient solution of (GMWD). Then there exists a point $(T^*, \tau^*, \lambda^*) \in \mathcal{D}$ such that $f(S^0) \leq f(T^*) + \lambda_{J_0}^{*\top} g_{J_0}(T^*)e$, which contradicts the weak duality Theorem 3. Therefore (S^0, τ^0, λ^0) is an efficient solution of (VP). □

Theorem 6 (Strong duality) *Suppose that* (j1) *and* (j2) *of Theorem 5 are satisfied. Then there are* $\tau^0 \in R^p$, $\tau^0 > 0$ *and* $\lambda^0 \in \mathbb{R}_+^m$ *such that* $(S^0, \tau^0, \lambda^0) \in \mathcal{D}$ *and the objective functions of* (*VP*) *and* (*GMWD*) *have the same values at* S^0 *and* (S^0, τ^0, λ^0)*, respectively.*
If problem (*VP*) *is* (ρ_0, ρ_0')*-semi strictly V-univex type I with* $\rho_0 + \rho_0' \geq 0$ *at all feasible solutions of* (*GMWD*) *according to the partition* $\{J_0, J_1, \ldots, J_k\}$*, with respect to* λ^0*, and conditions* (28)–(30) *of Theorem 4 are satisfied, then* $(S^0, \tau^0, \lambda^0) \in \mathcal{D}$ *is an efficient solution to* (*GMWD*).

Theorem 7 (Converse duality) *Suppose that*

(k1) $(T^0, \tau^0, \lambda^0) \in \mathcal{D}$ *with* $\tau^0 > 0$*;*
(k2) $T^0 \in \mathcal{P}$*;*
(k3) *problem* (*VP*) *is* (ρ, ρ')*-V-univex type I at* T^0*, with* $\sum_{i=1}^{p} \frac{\tau_i^0 \rho_i}{\alpha_i(S,T^0)} + \sum_{s=1}^{k} \frac{\rho_s'}{\beta_s(S,T^0)} \geq 0\ \ \forall\, S \in \mathcal{P}$*, according to the partition* $\{J_0, J_1, \ldots, J_k\}$*, with respect to* λ^0 *and some positive functions* α_i*,* $i \in \{1, \ldots, p\}$*, and* β_s*,* $s \in \{1, \ldots, k\}$*.*

Assume also that the functions φ_0 *and* φ_1 *have the properties*

$$\begin{cases} r < 0 \implies \varphi_0\,(r) < 0 \\ \varphi_0\,(0) \leqq 0 \\ r_1 \leqq r_2 \implies \varphi_0\,(r_1) \leqq \varphi_0\,(r_2)\,, \end{cases} \tag{31}$$

$$r \geqq 0 \implies \varphi_1\,(r) \geqq 0 \tag{32}$$

and

$$b_0(S, T^0) > 0,\ b_1(S, T^0) \geqq 0,\ \ \forall\, S \in \mathcal{P}. \tag{33}$$

Then T^0 *is an efficient solution of* (*VP*).
If, in addition, there exist positive numbers n_i, m_i *such that* $n_i < \alpha_i(S, T^0) < m_i$ *for all* $S \in \mathcal{P}$ *and* $i = 1, \ldots, p$*, then* T^0 *is properly efficient to* (*VP*).

Proof From hypothesis (k1), we have

$$\sum_{j \in J_s} \lambda_j^0 g_j(T^0) \geqq 0, \quad s \in \{1, \ldots, k\}\,. \tag{34}$$

By hypothesis (k3), using Definition 3, we have for any $S \in \mathcal{P}$ and i,

$$\begin{aligned} & b_0(S, T^0)\varphi_0\left[\psi_i\left(S, \lambda_{J_0}^0\right) - \psi_i\left(T^0, \lambda_{J_0}^0\right)\right] \\ & \quad \geqq \alpha_i(S, T^0)\left\langle D\psi_i\left(T^0, \lambda_{J_0}^0\right), I_S - I_{T^0}\right\rangle + \rho_i d^2(S, T^0), \end{aligned} \tag{35}$$

and for any s,

$$-b_1(S,T^0)\varphi_1\left[\sum_{j\in J_s}\lambda_j^0 g_j(T^0)\right] \geqq \beta_s(S,T^0)\sum_{j\in J_s}\lambda_j\left\langle Dg_j(T^0), I_S-I_{T^0}\right\rangle + \rho'_s d^2(S,T^0). \tag{36}$$

Since $\alpha_i>0$, $\beta_s>0$, for $\forall\, i,s$, and $\tau^0>0$, $\lambda^0\geqq 0$, it follows by (35), (36) and (k3) that

$$\begin{aligned}
&\sum_{i=1}^p \frac{\tau_i^0}{\alpha_i(S,T^0)} b_0(S,T^0)\varphi_0\left[\psi_i(S,\lambda_{J_0}^0)-\psi_i(T^0,\lambda_{J_0}^0)\right]\\
&\quad -\sum_{s=1}^k \frac{b_1(S,T^0)}{\beta_s(S,T^0)}\varphi_1\left[\sum_{j\in J_s}\lambda_j^0 g_j(T^0)\right]\\
&\quad \geqq \sum_{i=1}^p \tau_i^0\left\langle D\psi_i(T^0,\lambda_{J_0}^0), I_S-I_{T^0}\right\rangle + \sum_{j=1,\ j\notin J_0}^m \lambda_j^0\left\langle Dg_j(T^0), I_S-I_{T^0}\right\rangle\\
&\quad +\left(\sum_{i=1}^p \frac{\tau_i^0\rho_i}{\alpha_i(S,T^0)}+\sum_{s=1}^k\frac{\rho'_s}{\beta_s(S,T^0)}\right)d^2(S,T^0)\geqq 0
\end{aligned} \tag{37}$$

From (32), (34), and (37) it follows that for all $S\in\mathcal{P}$,

$$\sum_{i=1}^p \frac{\tau_i^0}{\alpha_i(S,T^0)} b_0(S,T^0)\varphi_0\left[\psi_i(S,\lambda_{J_0}^0)-\psi_i(T^0,\lambda_{J_0}^0)\right]\geqq 0. \tag{38}$$

Suppose that T^0 is not an efficient solution of (VP). Then there exists an $S'\in\mathcal{P}$ such that $f\left(S'\right)\leq F(T^0,\tau^0,\lambda^0)$, which implies, by using (31) and (33), that

$$\begin{aligned}
&\sum_{i=1}^p \frac{\tau_i^0}{\alpha_i(S',T^0)} b_0\left(S',T^0\right)\varphi_0\left[\psi_i(S',\lambda_{J_0}^0)-\psi_i(T^0,\lambda_{J_0}^0)\right]\\
&\quad\leqq \sum_{i=1}^p \frac{\tau_i^0}{\alpha_i(S',T^0)} b_0(S',T^0)\varphi_0\left[f_i(S')-\psi_i(T^0,\lambda_{J_0}^0)\right]<0.
\end{aligned} \tag{39}$$

Obviously, (38) and (39) are in contradiction. Therefore we get the conclusion of the theorem.

To establish the proper efficiency of T^0 for (VP), we define

$$M=(p-1)\max\left\{\frac{m_i\tau_j}{n_j\lambda_i}\ \middle|\ i,j\in\{1,\ldots,p\},\ i\neq j\right\}$$

and use (38) to get a contradiction. □

Also, following the same line as in the above proof we can get the next converse duality results by replacing the V-univexity with that of semi strictly pseudo V-univexity or the strictly pseudo quasi V-univexity hypotheses.

Theorem 8 (Converse duality) *Suppose that* (k1) *and* (k2) *of Theorem 7 are fulfilled and*

(k3′) *problem* (*VP*) *is* $\left(\rho_0,\rho'_0\right)$*-semi strictly pseudo V-univex type I in g, at* T^0*, with* $\rho_0+\rho'_0\geq 0$*, according to the partition* $\{J_0,J_1,\ldots,J_k\}$*, with respect to* τ^0,λ^0 *and some positive functions* α_i*,* $i\in\{1,\ldots,p\}$*,* β_s*,* $s\in\{1,\ldots,k\}$*.*

Assume also that the functions φ_0 *and* φ_1 *have the properties*

$$\varphi_0(r) \geqq 0 \implies r \geqq 0, \tag{40}$$

$$r \geqq 0 \implies \varphi_1(r) \geqq 0 \tag{41}$$

and

$$b_0(S, T^0) > 0,\ b_1(S, T^0) \geqq 0, \quad \forall\, S \in \mathcal{P}. \tag{42}$$

Then T^0 *is an efficient solution of* (*VP*).
If, in addition, there exist positive numbers n_i, m_i *such that* $n_i < \alpha_i(S, T^0) < m_i$ *for all* $S \in \mathcal{P}$ *and* $i \in \{1, \ldots, p\}$, *then* T^0 *is properly efficient to* (*VP*).

Theorem 9 (Converse duality) *Suppose that* (k1) *and* (k2) *of Theorem 7 are fulfilled and*

(k3″) *problem* (*VP*) *is* (ρ_0, ρ_0')*-strictly pseudo quasi V-univex type I at* T^0, *with* $\rho_0 + \rho_0' \geq 0$, *according to the partition* $\{J_0, J_1, \ldots, J_k\}$, *with respect to* τ^0, λ^0 *and some positive functions* α_i, $i \in \{1, \ldots, p\}$, β_s, $s \in \{1, \ldots, k\}$.

Assume also that the functions φ_0 *and* φ_1 *have the properties*

$$\begin{cases} r < 0 \implies \varphi_0(r) \leqq 0, \\ r_1 \leqq r_2 \implies \varphi_0(r_1) \leqq \varphi_0(r_2), \end{cases} \tag{43}$$

$$r \geqq 0 \implies \varphi_1(r) \geqq 0 \tag{44}$$

and

$$b_0(S, T^0) > 0,\ b_1(S, T^0) \geqq 0, \quad \forall\, S \in \mathcal{P}. \tag{45}$$

Then T^0 *is an efficient solution of* (*VP*).
If, in addition, there exist positive numbers n_i, m_i *such that* $n_i < \alpha_i(S, T^0) < m_i$ *for all* $S \in \mathcal{P}$ *and* $i = 1, \ldots, p$, *then* T^0 *is properly efficient for* (*VP*).

Remark 1 The advantage of Mond–Weir type dual problem over the primal one [15] is that the objective of the dual is the same as that of the primal, when $J_0 = \emptyset$, and more importantly, the convexity requirements for the dual problem can be relaxed. Moreover, the partition of the constraints index set makes possible to have fewer constraints in the dual problem (depending on the number of elements in the partition). For example, if the partition consists of two elements, J_0 and J_1, then the dual problem has only one constraint, whereas the primal problem has m constraints.

5 Some special classes of functions

We mentioned in the first section some references concerning the V-univexity or generalized V-univexity in the case of $\mathbb{R}^n$. Some of these references contain instances that certify the class of functions introduced there.

The aim of this section is to present how we can obtain some classes of functions defined in Sect. 2 starting from classes of functions defined on $\mathbb{R}^n$. Also, the way we proceed can be used to transform easily the examples (or combination of them) of V-univexity or generalized V-univexity type, given in the above mentioned references, into examples for optimization problems which involve set functions.

The following results apply only for the classes of functions that correspond to Definition 4. For the others, which correspond to Definitions 3, 5, 6 and 7, we can proceed similarly and for reasons of shortening we don't present them any more.

At the first, we reformulate a part of the Definition 4 for the $\mathbb{R}^n$ case.

Let us consider the multiobjective optimization problem

$$\begin{aligned}&\text{minimize } \tilde{f}(x) = \left(\tilde{f}_1(x), \ldots, \tilde{f}_p(x)\right)\\&\text{subject to } \tilde{g}(x) \leqq 0,\end{aligned} \qquad (\widetilde{\text{VP}})$$

where $\tilde{f} : X_0 \subseteq \mathbb{R}^n \to \mathbb{R}^p$, $\tilde{g} = (\tilde{g}_1(x), \ldots, \tilde{g}_m(x)) : X_0 \to \mathbb{R}^m$ are differentiable functions. Let $\tilde{X}_0 = \left\{x \in X_0 \mid \tilde{g}(x) \leqq 0\right\}$ be the set of feasible solutions and consider a partition $\{J_0, J_1, \ldots, J_k\}$ of the index set $\{1, 2, \ldots, m\}$. Let us consider

$$\tilde{\psi}_i\left(x, \lambda_{J_0}\right) = \tilde{f}_i(x) + \lambda_{J_0}^{\top} \tilde{g}_{J_0}(x), \quad i = 1, \ldots p,$$

and $\tilde{d} : X_0 \times X_0 \to \mathbb{R}$.

Definition 8 We say that the problem $(\widetilde{\text{VP}})$ is (ρ_0, ρ_0')-quasi V-univex type I according to the partition $\{J_0, J_1, \ldots, J_k\}$ if there exist positive real functions $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_p$ and $\tilde{\beta}_1, \ldots, \beta_k$ defined on $\mathbb{R}^n \times \mathbb{R}^n$, nonnegative functions $\tilde{b}_0$ and $\tilde{b}_1$, also defined on $\mathbb{R}^n \times \mathbb{R}^n$, $\varphi_0 : \mathbb{R} \to \mathbb{R}$, $\varphi_1 : \mathbb{R} \to \mathbb{R}$, such that for some vectors $\tau \in \mathbb{R}_+^p$ and $\lambda \in \mathbb{R}_+^m$ the following implications hold for $x, x^0 \in \tilde{X}_0$:

$$\begin{aligned}&\tilde{b}_0\left(x, x^0\right) \varphi_0 \left[\sum_{i=1}^{p} \tau_i \tilde{\alpha}_i(x, x^0) \left[\tilde{\psi}_i(x, \lambda_{J_0}) - \tilde{\psi}_i(x^0, \lambda_{J_0})\right]\right] \leqq 0\\&\Longrightarrow \sum_{i=1}^{p} \tau_i \sum_{t=1}^{n} \tilde{\psi}_i^t(x^0, \lambda_{J_0})(x_t - x_t^0) \leqq -\rho_0 \tilde{d}^2(x, x^0),\end{aligned} \tag{46}$$

and

$$\begin{aligned}&\tilde{b}_1\left(x, x^0\right) \varphi_1 \left[\sum_{s=1}^{k} \tilde{\beta}_s(x, x^0) \sum_{j \in J_s} \lambda_j \tilde{g}_j(x^0)\right] \leqq 0\\&\Longrightarrow \sum_{j=1,\ j \notin J_0}^{m} \lambda_j \sum_{t=1}^{n} \tilde{g}_j^t(x^0)(x_t - x_t^0) \leqq -\rho_0' \tilde{d}^2(x, x^0),\end{aligned} \tag{47}$$

where $\tilde{\psi}_i^t$ (respectively, $\tilde{g}_j^t$) is the tth partial derivative of $\tilde{\psi}_i$ (respectively, of $\tilde{g}_j$) with respect to x_t, the tth component of vector x.

Let us remark that for $S = (S_1, \ldots, S_n) \in \Gamma^{\mathrm{n}}$ and

$$G(S) = \theta\left(\langle h_1, I_{S_1}\rangle, \ldots, \langle h_n, I_{S_n}\rangle\right),$$

where $\theta : \mathbb{R}^n \to \mathbb{R}$ is a differentiable function, $h_1, \ldots, h_n \in L_1(X, \Gamma, \mu)$, we have

$$D_t G(S) = \theta^t\left(\langle h_1, I_{S_1}\rangle, \ldots, \langle h_n, I_{S_n}\rangle\right) h_t$$

where θ^t denotes the tth partial derivative of θ. Now we see that

$$\sum_{t=1}^{n}\left\langle D_t G(S^0), I_{S_t}-I_{S_t^0}\right\rangle=\sum_{t=1}^{n}\theta^t\left(\left\langle h_1, I_{S_1^0}\right\rangle,\ldots,\left\langle h_n, I_{S_n^0}\right\rangle\right)\left(\left\langle h_t, I_{S_t}\right\rangle-\left\langle h_t, I_{S_t^0}\right\rangle\right)$$

$$=\sum_{t=1}^{n}\theta^t\left(\left\langle h_1, I_{S_1^0}\right\rangle,\ldots,\left\langle h_n, I_{S_n^0}\right\rangle\right)\left\langle h_t, I_{S_t}-I_{S_t^0}\right\rangle$$

Now we can state the following result.

Proposition 1 *We suppose that* $(\widetilde{VP})$ *is* (ρ_0, ρ_0')*-quasi V-univex type I. Let us consider the problem* (VP) *with*

$$f_i(S)=u_i\left(\langle h_1, I_{S_1}\rangle,\ldots,\langle h_n, I_{S_n}\rangle\right),\quad i=1,\ldots,p,$$
$$g_j(S)=v_j\left(\langle h_1, I_{S_1}\rangle,\ldots,\langle h_n, I_{S_n}\rangle\right),\quad j=1,\ldots,m,$$

where $u_i:\mathbb{R}^n\to\mathbb{R}$, $i=1,\ldots,p$, *and* $v_j:\mathbb{R}^n\to\mathbb{R}$, $j=1,\ldots,m$, *are differentiable functions,* $h_1,\ldots,h_n\in L_1(X,\Gamma,\mu)$. *Then* (VP) *is* (ρ_0, ρ_0')*-quasi V-univex type I.*

Proof Let us consider that $(\widetilde{\text{VP}})$ is (ρ_0, ρ_0')-quasi V-univex type I. Then (46) and (47) hold. We shall prove that (VP) is also (ρ_0, ρ_0')-quasi V-univex type I by verifying that (3) and (4) hold.

Let $S^0\in\mathcal{P}$ and $S\in\mathcal{P}$ be such that

$$b_0(S,S^0)\varphi_0\left[\sum_{i=1}^{p}\tau_i\alpha_i(S,S^0)\left[\psi_i(S,\lambda_{J_0})-\psi_i(S^0,\lambda_{J_0})\right]\right]\leqq 0. \tag{48}$$

If we take

$$b_0(S,S^0)=\tilde{b}_0\left(\langle h_1, I_{S_1}\rangle,\ldots,\langle h_n, I_{S_n}\rangle;\left\langle h_1, I_{S_1^0}\right\rangle,\ldots,\left\langle h_n, I_{S_n^0}\right\rangle\right),$$

$$\alpha_i(S,S^0)=\tilde{\alpha}_i\left(\langle h_1, I_{S_1}\rangle,\ldots,\langle h_n, I_{S_n}\rangle;\left\langle h_1, I_{S_1^0}\right\rangle,\ldots,\left\langle h_n, I_{S_n^0}\right\rangle\right),$$

$$\psi_i\left(S,\lambda_{J_0}\right)=\tilde{\psi}_i\left(\langle h_1, I_{S_1}\rangle,\ldots,\langle h_n, I_{S_n}\rangle;\lambda_{J_0}\right),\quad i=1,\ldots,p,$$

and

$$d^2(S,S^0)=\tilde{d}^2\left(\langle h_1, I_{S_1}\rangle,\ldots,\langle h_n, I_{S_n}\rangle;\left\langle h_1, I_{S_1^0}\right\rangle,\ldots,\left\langle h_n, I_{S_n^0}\right\rangle\right),$$

then (48) becomes (46) with

$$\begin{aligned}x&=\left(\langle h_1, I_{S_1}\rangle,\ldots,\langle h_n, I_{S_n}\rangle\right),\\ x^0&=\left(\left\langle h_1, I_{S_1^0}\right\rangle,\ldots,\left\langle h_n, I_{S_n^0}\right\rangle\right)\end{aligned} \tag{49}$$

Using now the assumption that $(\widetilde{\text{VP}})$ is (ρ_0, ρ_0')-quasi V-univex type I, we get

$$\sum_{i=1}^{p}\tau_i\sum_{t=1}^{n}\tilde{\psi}_i^t\left(\left\langle h_1, I_{S_1^0}\right\rangle,\ldots,\left\langle h_n, I_{S_n^0}\right\rangle;\lambda_{J_0}\right)\left\langle h_t, I_{S_t}-I_{S_t^0}\right\rangle$$

$$\leqq -\rho_0\tilde{d}^2\left(\langle h_1, I_{S_1}\rangle,\ldots,\langle h_n, I_{S_n}\rangle;\left\langle h_1, I_{S_1^0}\right\rangle,\ldots,\left\langle h_n, I_{S_n^0}\right\rangle\right),$$

that is

$$\sum_{i=1}^{p} \tau_i \sum_{t=1}^{n} \left\langle D_t \psi_i \left(S^0, \lambda_{J_0}\right), I_{S_t} - I_{S_t^0} \right\rangle \leqq -\rho_0 d^2(S, S^0),$$

i.e., the implied condition in relation (3).

Now, to prove that the relation (4) in Definition 4 holds, we consider again $S^0 \in \mathcal{P}$, $S \in \mathcal{P}$, we choose

$$b_1(S, S^0) = \tilde{b}_1 \left(\langle h_1, I_{S_1} \rangle, \ldots, \langle h_n, I_{S_n} \rangle ; \left\langle h_1, I_{S_1^0} \right\rangle, \ldots, \left\langle h_n, I_{S_n^0} \right\rangle \right),$$
$$\beta_s(S, S^0) = \tilde{\beta}_s \left(\langle h_1, I_{S_1} \rangle, \ldots, \langle h_n, I_{S_n} \rangle ; \left\langle h_1, I_{S_1^0} \right\rangle, \ldots, \left\langle h_n, I_{S_n^0} \right\rangle \right),$$

for $s = 1, \ldots, k$, and we assume that

$$b_1(S, S^0)\varphi_1 \left[\sum_{s=1}^{k} \beta_s(S, S^0) \sum_{j \in J_s} \lambda_j g_j(S^0) \right] \leqq 0.$$

This relation is now equivalent to

$$\tilde{b}_1(x, x^0)\varphi_1 \left[\sum_{s=1}^{k} \tilde{\beta}_s(x, x^0) \sum_{j \in J_s} \lambda_j \tilde{g}_j(x^0) \right] \leqq 0,$$

where x, x^0 are defined by (49). Using now that ($\widetilde{\text{VP}}$) is $\left(\rho_0, \rho_0'\right)$-quasi V-univex type I, we obtain that

$$\sum_{j=1,\ j \notin J_0}^{m} \lambda_j \sum_{t=1}^{n} \tilde{g}_j^t(x^0)(x_t - x_t^0) \leqq -\rho_0' \tilde{d}^2(x, x^0),$$

i.e., (47) hold. Using now similar arguments as before, we get that this relation is equivalent to (4). Thus the proposition is proved. □

In the above Proposition we can take $X = \left\{(a, b) \in \mathbb{R}^2 \mid a^2 + b^2 \leqq 4\right\}$. Then $\Gamma = \{T \mid T \subseteq X\}$ is a σ-algebra. Let us consider μ be the Lebesgue measure. Then (X, Γ, μ) is a finite atomless measure space.

We see that for $r = 1, \ldots, n$, $h_r(a, b) = ra^2 + rb^2 + r \in L_1(X, \Gamma, \mu)$ and we can construct without difficulties classes of functions of those type given in Section 2 by taking different particular forms for $u_1, \ldots, u_p$ and $v_1, \ldots, v_m : \mathbb{R}^n \to \mathbb{R}$.

Starting from here we also can construct without difficulties particular dual problems.

References

1. Aghezzaf, B., Hachimi, M.: Generalized invexity and duality in multiobjective programming problems. J. Glob. Optim. **18**, 91–101 (2000)
2. Bătătorescu, A.: Optimality conditions involving V-type-I univexity and set-functions. Math. Rep. **7**(57), 1–11 (2005)
3. Begis, D., Glowinski, R.: Application de la méthode des éléments finis à l'approximation d'un problème de domain optimal. Méthodes de résolution des problemés Approches. Appl. Math. Optim. **2**, 130–169 (1975)
4. Cea, J., Gioan, A., Michel, J.: Quelques resultats sur l'identification de domaines. Calcolo **10**, 133–145 (1973)
5. Corley, H.W.: Optimization theory for n-set functions. J. Math. Anal. Appl. **127**, 193–205 (1987)

6. Corley, H.W., Roberts, S.D.: Duality relationships for a partitioning problem. SIAM J. Appl. Math. **23**, 490–494 (1972)
7. Corley, H.W., Roberts, S.D.: A partitioning problem with applications in regional design. Oper. Res. **20**, 1010–1019 (1982)
8. Hanson, M.A., Pini, R., Singh, C.: Multiobjective programming under generalized type-I invexity. J. Math. Anal. Appl. **261**, 562–577 (2001)
9. Jeyakumar, V., Mond, B.: On generalized convex mathematical programming. J. Austral. Math. Soc. Ser. B **34**, 43–53 (1992)
10. Mishra, S.K.: Generalized fractional programming problems containing locally subdifferentiable and ρ-univex functions. Optimization **41**, 135–158 (1997)
11. Mishra, S.K.: On multiple-objective optimization with generalized univexity. J. Math. Anal. Appl. **224**, 131–148 (1998)
12. Mishra, S.K., Rueda, N.G.: On univexity-type nonlinear programming problems. Bull. Allahabad Math. Soc. **16**, 105–113 (2001)
13. Mishra, S.K., Rueda, N.G., Giorgi, G.: Multiobjective programming under generalized type-I univexity. An. Univ. Bucureşti., Ser. Mat. **LII**(2), 207–224 (2003)
14. Mishra, S.K., Wang, S.Y., Lai, K.K., Shi, J.: New generalized invexity for duality in multiobjective programming problems involving n-set. Generalized convexity, generalized monotonicity and applications. In: Nonconvex Optim. Appl., vol. 77, pp. 321–339. Springer, New York (2005)
15. Mond, B., Weir, T. : Generalized concavity and duality. In: Schaible, S., Ziemba, T. (eds.) Generalized Concavity in Optimization and Economics, pp. 263–279. Academic Press, New York (1981)
16. Morris, R.J.T.: Optimal constrained selection of a measurable subset. J. Math. Anal. Appl. **70**, 546–562 (1979)
17. Neyman, J., Pearson, E.S.: On the problem of the most efficient tests of statistical hypotheses. Philos. Trans. Roy. Soc. London Ser. A **231**, 289–337 (1933)
18. Preda, V.: On duality of multiobjective fractional measurable subsets selection problems. J. Math. Anal. Appl. **196**, 514–525 (1995)
19. Preda, V.: Duality for multiobjective fractional programming problems involving n-set functions. In: Andreian Cazacu, C., Lehto, C.W.E., Rassias, T.M. (eds.) Analysis and Topology, pp. 569–583. World Scientific Publishing Co., River Edge, NJ (1998)
20. Preda, V., Bătătorescu, A.: On duality for minmax generalized B-vex programming involving n-set functions. J. Convex Anal. **9**(2), 609–623 (2002)
21. Preda, V., Stancu-Minasian, I.M.: Optimality and Wolfe duality for multiobjective programming problems involving n-set functions. In: Konnov, I.V., Luc, D.T., Rubinov, A.M. (eds.) Lecture Notes in Economics and Mathematical Systems, vol. 502, pp. 349–361. Springer-Verlag, Berlin (2001)
22. Preda, V, Stancu-Minasian, I.M., Koller, E.: On optimality and duality for multiobjective programming problems involving generalized d-type-I and related n-set functions. J. Math. Anal. Appl. **283**, 114–128 (2003)
23. Stancu-Minasian, I.M.: Fractional Programming: Theory, Methods and Applications. Kluwer Academic Publishers, Dordrecht (1997)
24. Stancu-Minasian, I.M., Preda, V.: Optimality conditions and duality for programming problems involving set and n-set functions—a survey. J. Stat. Manage Syst. **5**, 175–207 (2002)
25. Venkateswara, R.L.: Generalized pseudo univexity and duality in mathematical programming. Indian J. Pure Appl. Math. **32**(5), 717–725 (2001)
26. Yuan, D., Chinchuluun A., Liu, X.P., Pardalos, P.M.: Optimality conditions and duality for multiobjective programming involving (C, α, ρ,d) type-I functions, In: Konnov, I.V., Luc, D.T. and Rubinov, A.M. (eds.) Lecture Notes in Economics and Mathematical Systems, vol. 573, pp. 73–87. Springer (2007)
27. Zalmai, G.J.: Sufficiency criteria and duality for nonlinear programs involving n-set functions. J. Math. Anal. Appl. **149**, 322–338 (1990)
28. Zalmai, G.J.: Optimality conditions and duality for multiobjective measurable subset selection problems. Optimization **22**, 221–238 (1991)